Journal of Statistical Physics, Vol. 58, Nos. 1/2, 1990

The Heisenberg *XXZ* Hamiltonian with Dzyaloshinsky–Moriya Interactions

F. C. Alcaraz[1] **and W. F. Wreszinski**[2]

Received June 27, 1989; revision received September 1, 1989

The quantum Heisenberg chain with Dzyaloshinsky-Moriya interactions is solved by relating it to the *XXZ* Hamiltonian with a certain type of boundary conditions. Several properties of the ground state are derived which agree with the intuition derived from related soluble classical models. Implications to the model of known results from the theory of conformal invariance, as well as generalizations to higher spin, are briefly discussed.

KEY WORDS: Quantum Heisenberg chain; Dzyaloshinsky–Moriya interactions; Bethe ansatz; boundary conditions; conformal invariance.

1. INTRODUCTION AND SUMMARY

Lattice systems wit competitive interactions have aroused great interest, both theoretical and experimental, in the last decade[1] Of special importance are uniaxially modulated magnetic structures, where the expectation value of the spin varies periodically only along one crystallographic direction and in which, basically, two types of arrangement may be generated: (a) an antiphase structure and (b) a spiral or helicoidal structure. In case (a) the spins are of Ising type, i.e., have only one component different from zero, as in the so-called ANNNI model (see ref. 1 and references given there). In case (b) the spins have more than one nonzero component and neighbor spins make a fixed angle φ with one another in a certain region of temperature and coupling constant.

In this paper we consider quantum versions of case (b). In order to describe our motivation, consider the classical *XY* chain with

[1] Departamento de Física, Universidade Federal de São Carlos, 13560 São Carlos, SP, Brazil.
[2] Instituto de Física, Universidade de São Paulo, 01498 São Paulo, SP, Brazil.

Dzyaloshinsky–Moriya[2] (DM) interactions. The Hamiltonian for N spins is given by

$$H_N = -J_1 \sum_{i=1}^{N} \mathbf{S}_i \cdot \mathbf{S}_{i+1} + J_2 \sum_{i=1}^{N} (\mathbf{S}_i \times \mathbf{S}_{i+1}) \cdot \hat{\mathbf{z}} \tag{1.1}$$

with $J_1 \geqslant 0$, $J_2 \geqslant 0$ (other cases are related by canonical transformations), and where the boundary conditions (b.c.) are left open for the time being. The above classical spins are unit vectors in the plane, $\mathbf{S}_i \equiv S_i^x, S_i^y)$, with $S_i^x = \cos\alpha_i$ and $S_i^y = \sin\alpha_i$, $i = 1,..., N$, the cross denotes vector product, and $\hat{\mathbf{z}}$ is a unit vector along the z axis. Clearly, the first term in (1.1) tends to render neighbor spins parallel, while the second has the effect of turning them perpendicular to one another. The resulting ground state has a spiral structure:

$$H_N = -\frac{J_1}{\cos\phi} \sum_{i=1}^{N} \cos(\alpha_i - \alpha_{i+1} - \phi) \tag{1.2}$$

where ϕ is defined by

$$\phi = \arctan J_2/J_1, \qquad \phi \in [0, \pi/2] \tag{1.3a}$$

By the above choice of ϕ, $(J_1/\cos\phi) \geqslant 0$ and the ground state is given by

$$\alpha_i - \alpha_{i+1} = \phi \tag{1.3b}$$

as long as

$$\alpha_{N+1} = p(\alpha_1 + \phi), \qquad p = 0, 1 \tag{1.4}$$

where $p = 0$ corresponds to free b.c. and $p = 1$ to the angle-dependent b.c. of Section 2. We therefore see, even in this elementary example, the essential role played by the boundary condition (1.4). Note that the classical ground state has helical structure however small J_2, showing already a fundamental qualitative difference between systems with DM interaction and those models of ANNNI type[1] where competition is introduced through a next-nearest-neighbor interaction of antiferromagnetic type. In fact, the corresponding classical XY chain is also soluble[3,4] and it is found that the ground state remains ferromagnetic if the ratio of next-nearest antiferromagnetic coupling to nearest-neighbor ferromagnetic coupling is sufficiently small.

The quantum version of (1.1)

$$H_N(-J, \Delta) = \sum_{i=1}^{N} [J(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + \Delta(S_i^x S_{i+1}^y - S_{i+1}^x S_i^y)] \tag{1.5}$$

with periodic boundary conditions, $J = -1$ (antiferromagnetic case), and $\Delta = J_2/J_1 = -J_2$ was solved in refs. 5 and 6 by a Jordan–Wigner transformation. Correlation functions in the ground state $|\psi_0\rangle$ such as

$$C\mathscr{F}_Q \equiv \langle \psi_0 |\mathbf{S}_i \cdot \mathbf{S}_{i+1}| \psi_0 \rangle \tag{1.6}$$

may be computed through the use of Wick's theorem and the result is [ref. 5, formula (7.15), p. 282]

$$C\mathscr{F}_Q = -\frac{1}{\pi}(\Delta^2+1)^{-1/2} = \frac{1}{\pi}\cos\arctan(-\Delta) \tag{1.7}$$

taking $\arctan(-\Delta) \in [0, \pi/2)$ as in the classical case. The authors of ref. 5 did not compare their results with the classical case (!), but we see that, apart from the factor $1/\pi$ (which is also present if $\Delta = 0$), (1.7) and (1.3) are equal, showing that the ground state is qualitatively the same as the classical one; the discrepancy arises from the degeneracy of the classical ground state. Indeed, (1.7) may be written

$$C\mathscr{F}_Q = \frac{C\mathscr{F}_{\mathrm{cl}}}{\pi = (\text{area of region where } \mathbf{S}_1 \text{ may vary})}$$

where $C\mathscr{F}_{\mathrm{cl}} \equiv \cos\arctan(-\Delta)$ is the classical correlation function. Clearly, fixing the position of $\mathbf{S}_1$ and the boundary condition determines uniquely the ground-state configuration.

It is a basic question, both from the conceptual and the experimental point of view (due to "quantum crossover" effects at low temperatures), whether quantum ground (and low-lying) states differ qualitatively from their classical counterparts, and, in general, what such states look like. In this paper we take a step in this direction by solving the quantum Heisenberg chain with DM interactions

$$H_N = H_N(2, 2\Delta) - 2\delta \sum_{i=1}^{N} S_i^z S_{i+1}^z \tag{1.8}$$

where $H_N(\alpha, \beta)$ was defined in (1.5). The results are presented in Section 2, which is divided into five parts for clarity. The (Bethe-ansatz) solution and the corresponding phase diagram are presented and discussed in Section 2.1. There, again, a crucial role is played by the boundary conditions. We study both free and toroidal boundary conditions [analogous to $p = 1$ in (1.5)] in Sections 2.2 and 2.3, respectively. In Section 2.2 (free b.c.) the ground-state energy per unit volume $e_{\Delta,\delta}$ is derived, as well as some of its properties: in particular, the special case $\delta = 0$ of ref. 5 is obtained, and the inequality $e_{\Delta,\delta} \leqslant e_{0,\delta}$, obvious for $\delta = 1$ (isotropic ferromagnet), is derived in general. For a special value of $\Delta \neq 0$ it is also shown that $e_{\Delta,-1} < e_{0,-1}$. In Section 2 (toroidal b.c.) we discuss the excitation spectrum. In Section 2.4 we briefly discuss some implications of known results in the theory of conformal invariance[16] to the present model, as well as extensions to models of higher spin. Section 2.5 is devoted to conclusions and open problems.

2. RESULTS FOR THE QUANTUM HEISENBERG CHAIN WITH DZYALOSHINSKY–MORIYA INTERACTIONS

In this section we will solve the quantum anisotropic Heisenberg chain (or XXZ chain) with Dzyaloshinsky–Moriya interactions (1.8), which we rewrite now in terms of Pauli matrices $\sigma_i^x = 2S_i^x$,

$$H_N^{\mathrm{DM}}(\Delta, \delta) = H_N^{XXZ}(\delta) + V_N^{\mathrm{DM}}(\Delta) \tag{2.1a}$$

where

$$H_N^{XXZ}(\delta) \equiv -\frac{1}{2} \sum_{i=1}^{N} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \delta \sigma_i^z \sigma_{i+1}^z) \tag{2.1b}$$

is the XXZ quantum Hamiltonian, with anisotropy δ, and

$$V_N^{\mathrm{DM}}(\Delta) \equiv -\frac{\Delta}{2} \sum_{i=1}^{N} (\sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^y) \tag{2.1c}$$

is the Dzyaloshinsky–Moriya interaction.

2.1. Solution and Phase Diagram

The solution of Hamiltonian (2.1) through the Bethe ansatz is obtained by relating it to the XXZ chain with a certain type of b.c.

In order to proceed, let us introduce the following matrices, located at the sites of the finite (size N) chain

$$\sigma_j^{\pm 1} = \tfrac{1}{2}(\sigma_j^x \pm i\sigma_j^y), \qquad \sigma_j^0 = \sigma_j^z, \qquad j = 1, 2, \ldots, N \tag{2.2}$$

The commutation relations of these matrices are invariant under the general canonical transformation

$$\sigma_j'^m = \sum_{n=-1}^{1} A^{m,n} \sigma_j^n, \qquad j = 1, 2, \ldots, N \tag{2.3a}$$

where $A^{m,n}$ are (3×3) matrices forming the $O(2)$ group, i.e.,

$$A \in \{G(\theta)\cdot C^{\alpha} \mid \theta \in [0, 2\pi), \alpha = 0, 1\} \tag{2.3b}$$

where

$$G(\theta) = \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.3c}$$

In terms of these variables (2.1) may be rewritten as

$$H_N^{\mathrm{DM}}(\Delta, \delta) = -\frac{1}{\cos\phi}\left[\sum_{i=1}^{N-1} (e^{i\phi}\sigma_i^{+1}\sigma_{i+1}^{-1} + e^{-i\phi}\sigma_i^{-1}\sigma_{i+1}^{+1} + \delta\cos\phi\, \sigma_i^0\sigma_{i+1}^0\right] + H_S^{p,\Omega} \tag{2.4a}$$

where

$$-\pi/2 < \phi \equiv \tan^{-1}\Delta < \pi/2 \tag{2.4b}$$

and $H_S^{p,\Omega}$ specifies the boundary condition. This Hamiltonian, apart from the surface term $H_S^{p,\Omega}$, commutes with the z component of the total spin operator $S_z = \sum_{i=1}^{N} \sigma_i^z$, which implies that, in the σ^z basis, the Hilbert space may be separated into block disjoint sectors labeled by the eigenvalues $n = 0, \pm 1, \pm 2,\ldots$ of S_z. The most general boundary conditions compatible with this symmetry are

$$\sigma_{N+1}^{\pm 1} = pe^{\pm i\Omega}\sigma_1^{\pm 1}, \qquad \sigma_{N+1}^0 = p\sigma_1^0 \tag{2.5}$$

where $0 \leqslant \Omega < 2\pi$ and $p = 0, 1$. The angle Ω specifies a rotation, around the z axis, of the spin operator $\boldsymbol{\sigma}_{N+1}$ with respect to $\boldsymbol{\sigma}_1$. The case of free boundaries corresponds to $p = 0$ and the periodic chain to $p = 1$ and $\Omega = 0$. Using these boundary conditions, the surface term $H_S^{p,\Omega}$ in (2.4a) is given by

$$H_N^{p,\Omega} = -\frac{p}{\cos\phi}(e^{i(\phi-\Omega)}\sigma_N^{+1}\sigma_1^{-1} + e^{-i(\phi-\Omega)}\sigma_N^{-1}\sigma_1^{+1} + \delta\cos\phi\sigma_N^0\sigma_1^0) \tag{2.6}$$

By making the canonical transformation [see (2.3)]

$$\sigma_j^{\pm 1} \to \sigma_j^{\pm 1} e^{\pm i(2j-3)\phi/2}, \qquad \sigma_j^0 \to \sigma_j^0 \tag{2.7}$$

the Hamiltonian takes the simple form

$$H_N^{\mathrm{DM}}(\Delta, \delta) = \frac{1}{\cos\phi} H_N^{XXZ}(\delta\cos\phi) \tag{2.8a}$$

where

$$\begin{aligned} H_N^{XXZ}(\delta\cos\phi) = -\sum_{i=1}^{N} (\sigma_i^{+1}\sigma_{i+1}^{-1} + \sigma_i^{-1}\sigma_{i+1}^{+1} \\ + \delta\cos\phi\sigma_i^0\sigma_{i+1}^0) - p(e^{i(N\phi-\Omega)}\sigma_N^{+1}\sigma_1^{-1} \\ + e^{-i(N\phi-\Omega)}\sigma_N^{-1}\sigma_1^{+1} + \delta\cos\phi\,\sigma_N^0\sigma_1^0) \end{aligned} \tag{2.8b}$$

is, from (2.2) and (2.1b), the quantum XXZ Hamiltonian with anisotropy $\delta\cos\phi$ and boundary condition [see (2.5)]

$$\sigma_{N+1}^{\pm 1} = pe^{\mp i(N\phi-\Omega)}\sigma_1^{\pm 1}, \qquad \sigma_{N+1}^0 = p\sigma_1^0 \tag{2.8c}$$

The result (2.8) shows us that the effect of the introduction of the Dzyaloshinsky–Moriya interaction in the XXZ quantum chain is the change of the anisotropic constant $\delta \to \delta\cos\phi = \delta/(1+\Delta^2)^{1/2}$ and of the boundary conditions $\Omega \to \Omega - N\phi$ [see (2.5) and (2.9)]. From the usual arguments, in the thermodynamic limit ($N \to \infty$) the boundary condition does not affect the critical behavior and consequently the Hamiltonian $H_\infty^{\mathrm{DM}}(\Delta, \delta)$ will have the same critical properties as the XXZ Hamiltonian $H_\infty^{XXZ}(\delta\cos\phi)$.

Hence, the phase diagram of the Hamiltonian $H_\infty^{\mathrm{DM}}(\Delta, \delta)$ may be obtained from the known[9] critical properties of the XXZ chain. Figure 1 shows this phase diagram where a critical phase C (massless) is separated from the massive phases A and F by the lines

$$\delta = \pm(1+\Delta^2)^{1/2} \tag{2.9}$$

From the exact results of the XXZ quantum chain the critical exponents in this massless phase change continuously with the coupling constants Δ and δ. We also show in Fig. 1 a dashed line where all the critical exponents are constant. The particular line $\delta = 0$ shows us that the Dzyaloshinsky–Moriya interaction does not modify the critical behavior of the $x-y$ model.

Let us now consider the effects of the boundary condition explicitly.

2.2. Free Boundary Condition ($p = 0$)

In this case the relations (2.8) and (2.9) show that the Hamiltonian $H_N^{\mathrm{DM}}(\Delta, \delta)$ is totally equivalent to the XXZ quantum chain with free ends

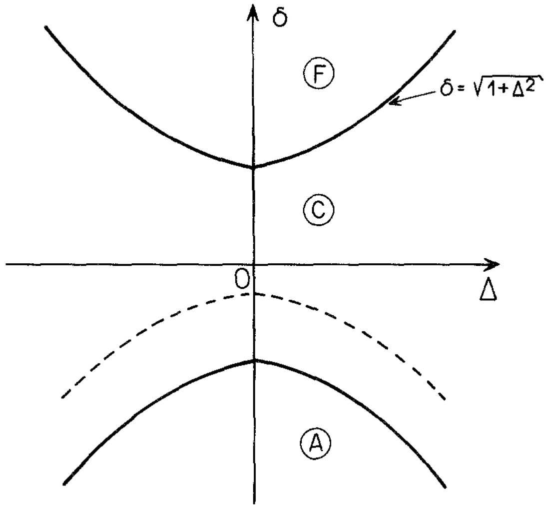

Fig. 1. The phase diagram of $H_\infty^{\mathrm{DM}}(\Delta, \delta)$. The critical lines $\delta = \pm(1+\Delta^2)^{1/2}$ separate the massless phase C from massive phases A and F. In the critical phase C the exponents change continuously in the space (Δ, δ). On the dashed line as well on the line $\delta = 0$ (xy model) the critical exponents are constant.

and anisotropy constant $\delta/(1+\Delta^2)^{1/2}$. In this case the N dependence of the phase in (2.9) does not occur and the thermodynamic limits of H_N^{DM} and H_N^{XXZ} are identical with the phase diagram of Fig. 1. The XXZ chain with free ends, in the same way as in the periodic case,[10] is also exactly integrable through the Bethe ansatz[11,12] and the properties of $H_N^{\mathrm{DM}}(\Delta, \delta)$ follow from the relations (2.8). For example, the ground-state energy per article in the critical phase $-1 \leqslant [-\delta/(1+\Delta^2)^{1/2} = \cos\gamma] \leqslant 1$ is given by[11,12]

$$e_\infty(\delta, \Delta) = -\tfrac{1}{2}(1+\Delta^2)^{1/2}\,(\cos\gamma + 4\sin^2\gamma \int_0^\infty dx \\ \times \{\cosh(\pi x)[\cosh(2\gamma x) - \cos\gamma]\}^{-1}) \qquad (2.10)$$

The expression (2.10) has interesting consequences. In particular, for $\delta = 0$, (2.10) yields

$$e_\infty(0, \Delta) = -\frac{1}{2}(1+\Delta^2)^{1/2} \cdot 4 \int_0^\infty dx[\cosh(\pi x)]^{-2} = -\frac{2}{\pi}(1+\Delta^2)^{1/2}$$

which agrees with (4.29) of ref. 5 except for the factor 2, which follows from the fact that (2.1) equals (for $\delta = 0$) twice the Hamiltonian of ref. 5. This is especially significant because the result of ref. 5 was obtained by an entirely different method (the Jordan–Wigner transformation).

Since $V_N^{\mathrm{DM}}(\Delta)$ defined by (2.1c) has zero expectation value in the ferromagnetic ground state (all spins "up" or "down"), it is clear that $e_\infty(1, \Delta) \leqslant e_\infty(1, 0)$. In general the inequality is not obvious, but it follows from a trick used by Affleck and Lieb[9] in a different context:

Lemma $e_\infty(\delta, \Delta) \leqslant e_\infty(\delta, 0)$.

Proof. Let $H_N^{XXZ}(\delta)$ and $V_N^{\mathrm{DM}}(\Delta)$ be defined by (2.1b) and (2.1c), respectively, with *periodic b.c.*, and define

$$A_N \equiv \frac{1}{2} \sum_{i=1}^{N} i\sigma_i^z$$

Then

$$[A_N, H_N^{XXZ}(\delta)] = i\Delta V_N^{\mathrm{DM}}(\Delta)$$

Hence, $(\Psi, V_N^{\mathrm{DM}}(\Delta)\, \Psi) = 0$ if Ψ is any eigenstate of $H_N^{XXZ}(\delta)$, and the result follows from the variational principle and the fact that e_∞ is independent of b.c. $\square$

For $\delta = -1$,

$$e_\infty(-1, 0) = \tfrac{1}{2} - 2 \ln 2 \cong -0.8862944$$

(isotropic antiferromagnet[9]). Let, now, $\gamma = \pi/6$, $\delta/(1+\Delta^2)^{1/2} = -\sqrt{3}/2$; putting $\delta = -1$, $(1+\Delta^2)^{1/2} = 2/\sqrt{3}$ or $\Delta = 1/\sqrt{3} \cong 0.5780347$. By Table I of ref. 8,

$$\begin{aligned} e_\infty\left(-1, \frac{1}{\sqrt{3}}\right) &= \left(-\frac{1}{\pi} - \frac{11}{12\sqrt{3}}\right)\frac{2}{\sqrt{3}} = -\frac{2}{\pi\sqrt{3}} - \frac{11}{18} \\ &\cong -0.9790984 \end{aligned}$$

We see therefore that in this special case the strict inequality

$$e_\infty(\delta, \Delta) < e_\infty(\delta, 0)$$

obtains. The above inequality may be expected in general by analogy with the classical result [see (1.2)].

2.3. Toroidal Boundary Condition ($p = 1$)

We consider now the case where $p = 1$ and $0 < \Omega < 2\pi$ in (2.4)–(2.6). The results (2.8) indicate now that $H_N^{\mathrm{DM}}(\Delta, \delta)$ is equivalent to the XXZ

chain with anisotropy $\delta/(1+\Delta^2)^{1/2}$ and boundary condition $\sigma_{N+1}^{\pm 1} = e^{\mp i(N\phi - \Omega)}\sigma_1^{\pm 1}$ and σ_{N+1}^0. With such toroidal boundary condition this last model is still soluble, by using the Bethe ansatz.[7,8] Using these results, the eigenenergies of the sector $n = 0, \pm 1, \pm 2,...$ of $H_N^{\mathrm{DM}}(\Delta, \delta)$ are given by

$$E_n = -\frac{1}{2}(1+\Delta^2)^{1/2}\left\{N\cos\gamma + 4\sin^2\gamma \sum_{j=1}^{N/2-n} [\cosh(2\gamma\lambda_j) - \cos\gamma]^{-1}\right\} \tag{2.11}$$

where $\delta\cos\phi = \delta/(1+\Delta^2)^{1/2} = \cos\gamma$ and $\{\lambda_j, j = 1, 2,..., N/2 - n\}$ are the roots of the Bethe ansatz equations

$$\left[\frac{\sinh\gamma(\lambda_j - i/2)}{\sinh\gamma(\lambda_j + i/2)}\right]^N e^{i(N\phi - \Omega)} = -\prod_{k=1}^{N/2-n}\left[\frac{\sinh\gamma(\lambda_j - \lambda_k - i)}{\sinh\gamma(\lambda_j - \lambda_k + i)}\right] \tag{2.12}$$
$$j = 1, 2,..., N/2 - n$$

The effect of the boundary condition is the appearance of the phase $e^{i(N\phi - \Omega)}$ in the left-hand side of (2.12). The zeros $\{\lambda_i\}$ of these equations, corresponding to the eigenstate with lowest energy in the sector $n = 0, \pm 1, \pm 2,...$, are real numbers and we can easily convert the complex equations (2.12) into real ones. In this case these eigenenergies for large, but finite lattice size N can be estimated by using standard methods.[13] These energies E_0^n in the region $-1 \leqslant \cos\gamma \leqslant 1$ are given by

$$\frac{E_0^n}{N} = e_\infty - \frac{\pi}{6}\frac{\zeta}{N^2}(1 - 12\, X_{n,\Psi_N}) + o(N^{-2}) \tag{2.13a}$$

where

$$\zeta = \frac{\pi\sin\gamma}{\gamma}(1+\Delta^2)^{1/2}; \qquad X_{n,\Psi_N} = n^2 X + \frac{\Psi_N^2}{4X} \tag{2.13b}$$

$$X = (\pi - \gamma)/2\pi \tag{2.13c}$$

$$0 \leqslant \Psi_N \equiv \left(\frac{N\phi - \Omega}{2\pi}\right) \bmod 1 < 1 \tag{2.13d}$$

and e_∞ is given by (2.11). The above results show us that all these states degenerate as $N \to \infty$, indicating that the theory is gapless for $-1 \leqslant \cos\gamma \leqslant 1$, or in the critical region of Fig. 1.

Relation (2.13) has two consequences. First, for $\gamma \neq \pi[\delta/(1+\Delta^2)^{1/2} \neq 1]$ it lies in the sector $n = 0$. This agrees with classical intuition, at least in the isotropic antiferromagnetic case ($\delta = 1$): the corresponding minimal energy classical configuration has a helical structure corresponding to (1.2) in the x–y plane, with z components of neighboring spins antiparallel. Second, if $\phi/2\pi = p/q$, $p, q \in \mathbb{Z}$, $q \neq 0$, it is possible to choose sequences of lattice sizes $S_r^{(i)} = \{N_r^{(i)} = qi + r; i = 0, 1, 2,...\}$, with $r = 0, 1,..., q-1$, such that $\psi_{N_r^{(i)}}$, given by (2.13d), is independent of i and equals $\psi_{N_r^{(i)}} = (r\phi - \Omega)/2\pi$. By (2.13a) and (2.13b), if the boundary angle is then chosen as $\Omega = r\phi$, the energies E_0^n (and in particular the ground-state energy, corresponding to $n = 0$) are minimized. No such choice is possible if $\phi/2\pi$ is irrational. Again, this result is intuitive if one thinks of the classical helicoidal ground state of (1.2).

2.4. Implications of Conformal Invariance and Higher Spin Models

The assumption of conformal invariance of critical models in $(1+1)$ dimensions produces remarkable relations[15] between the finite-size corrections of the eigenspectrum of finite-size Hamiltonians and the scaling dimensions of operators governing the critical behavior. Relations (2.13) are in complete agreement with these predictions whenever $\phi/2\pi$ is rational and we use the sequences (2.14b) in the $N \to \infty$ limit. These predictions also state that the finite-size corrections of the vacuum energy of the finite quantum Hamiltonians, which are normally achieved when periodic boundaries are imposed, are proportional to the central charge c, or conformal anomaly of the conformal theory.[14] As remarked in the previous section, (2.13) shows us that the true vauum, for $\phi/2\pi = p/q$ and the lattice-size sequences S_r [see (2.14b)], is obtained by choosing in (2.5) the boundary angle $\Omega = \phi r \pmod{2\pi}$, which is not periodic in general. Choosing this boundary condition for the Hamiltonian $H_N^{\mathrm{DM}}(\Delta, \delta)$, its eigenspectrum for the sequence S_r is totally equivalent to the XXZ chain with periodic boundaries and anisotropy $\gamma = \cos^{-1}[-\delta/(1+\Delta^2)^{1/2}]$. From the equivalence (2.8) and the known spectroscopic calculations of the XXZ chain with general boundary conditions,[7,8,16] we conclude that $H_\infty^{\mathrm{DM}}(\Delta, \delta)$ in its critical phase, whenever $\phi/2\pi$ is rational, is governed by a theory with conformal central charge $c = 1$ and operators satisfying a $U(1)$ Kac–Moody algebra. The whole operator content can be promptly derived.

It is an interesting result[16] that the whole operator content of the minimal models[17] with conformal anomaly $c = 1 - 6/m(m+1)$, $m = 3, 4,...$, may be derived from the XXZ chain with several boundary conditions. The relations (2.8) imply that the same results can be obtained from the $H^{\mathrm{DM}}(\Delta, \delta)$ with several values of the couplings and boundary conditions. In particular,[7,8,16] these results imply that the ground-state energy of the periodic $(\Omega = 0)$ N_i-sites Hamiltonian $H_{N_i}^{\mathrm{DM}}(\Delta, 1)$, where $N_i =$

$2(m+1)\,i+2$, $i=0, 1, 2,\ldots$, and $\Delta=\tan(\pi/m+1)$ corresponds exactly to the ground-state energy of an N_i-site Hamiltonian describing the above minimal model when $m=3, 4,\ldots$. In the case $m=3$ and 4, where the corresponding minimal models are the Ising and three-state Potts models, these correspondences between the finite chains are exact.

The results derived in Sections 2.1–2.3 may also be extended to generalizations of Heisenberg models with arbitrary spin S (1, 3/2, 2,...). Let us consider the arbitrary spin-S quantum Hamiltonian

$$H_S=\sum_i Q(\mathbf{S}_i\cdot\mathbf{S}_{i+1}) \tag{2.15}$$

where $Q(x)$ is an arbitrary polynomial of degree $2S$ and $\mathbf{S}_i\equiv(S_i^x, S_i^y, S_i^z)$ are the $SU(2)$ spin-S matrices. We now introduce the Dzyaloshinsky–Moriya interactions in the following way:

$$H_S^{\mathrm{DM}}(\Delta)=\sum_i Q[(1+\Delta^2)^{1/2}\,(S_i^xS_{i+1}^x+S_i^yS_{i+1}^y+\Delta(\mathbf{S}_i\times\mathbf{S}_{i+1})\cdot z+S_i^zS_{i+1}^z] \tag{2.16}$$

We may show in a similar way as in Sections 2.1–2.3 that (2.16) and (2.15) are exactly related. The Hamiltonian (2.1) with boundary condition specified by p and Ω [see (2.5)] is exactly equal to the Hamiltonin (2.15) with boundary condition specified by p'/p and $\Omega'=\Omega-N\phi$. If we choose in (2.15) the particular polynomial $Q(x)$ introduced by Babujian and Takhtajan,[18] which is soluble through the Bethe ansatz, then (2.16) will be also an exactly integrable Hamiltonian. The Bethe ansatz equations, in this case, for (2.16) with boundary condition $p=1$ and $0\leqslant\Omega\leqslant 2\pi$ [see (2.5)] are[19]

$$\left(\frac{\lambda_i-iS}{\lambda_i+iS}\right)e^{i(N\phi-\Omega)}=-\prod_{k=1}^{NS-n}\left(\frac{\lambda_i-\lambda_j-i}{\lambda_i-\lambda_j+i}\right)$$

where $n=\sum_{i+1}^N S_i^z$. This is the generalization of (2.12) in the isotropic case $(\gamma\to 0)$. It is also easy to derive an exactly integrable model which corresponds to the anisotropic generalization of (2.12).[20]

The same physical analysis of Sections 2.1–2.3 is also true for these general models, which are also critical.[18–20]

2.5. Conclusions and Open Problems

Although we have shown that some properties of the quantum model agree qualitatively with the intuition derived from classical models, several problems remain. In particular, quantum correlation functions such as (1.6) remain to be studied and might still show qualitative differences with respect to their classical counterparts. Also, the thermodynamics remains to be studied, in particular whether the low-temperature behavior of the specific heat is different according to whether $\phi/2\pi$ is rational or irrational, in analogy to some results for classical models.[21]

A second open problem is to find (at least partially) soluble models in higher dimensions. A standard approach to this problem is to formulate a mean-field version of the model, and then study a (Bethe–Peierls–Kikuchi) "expansion" around the mean-field solution. It does not seem easy, however, to find an interesting mean-field version, possibly because of the essentially local nature of DM interactions.

NOTE ADDED IN PROOF

The interesting question referred to in the text as the second consequence of (2.13) (see the end of section 2.3) has been studied in great detail and in a much wider context by Woynarovich, Eckle and Truong, *J. Phys. A: Math. Gen.* **22**:4027 (1989).

REFERENCES

1. P. Bak, *Rep. Progr. Phys.* **45**:587 (1982).
2. I. E. Dzyaloshinsky, *J. Phys. Chem. Solids* **4**:241 (1958); T. Moriya, in *Magnetism*, G. T. Rado and H. Suhl, eds. (Academic Press, New York, 1963).
3. A. Yoshimori, *J. Phys. Soc. Japan* **14**:807 (1959).
4. J. Villain, *J. Phys. Chem. Solids* **23**:287 (1962).
5. Th. J. Siskens, H. W. Capel, and K. J. F. Gaemers, *Physica* **79A**:259 (1975).
6. J. L. van Hemmen and G. Vertogen, *Physica* **81A**:391 (1975).
7. F. C. Alcaraz, M. N. Barber, and M. T. Batchelor, *Phys. Rev. Lett.* **58**:771 (1987).
8. F. C. Alcaraz, M. N. Barber, and M. T. Batchelor, *Ann. Phys.* **182**:280 (1988).
9. I. Affleck and E. H. Lieb, *Lett. Math. Phys.* **12**:57 (1986).
10. C. N. Yang and C. P. Yang, *Phys. Rev. B* **150**:321 (1966); *Phys. Rev. B* **150**:327 (1966).
11. M. Gaudin, *La Fonction d'onde de Bethe* (Masson, Paris, 1983).
12. F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter, and G. R. W. Quispel, *J. Phys. A* **20**:6397 (1987).
13. F. Woynarovich and H. P. Eckle, *J. Phys. A* **20**:L97, L443 (1987); K. Karowski, *Nucl. Phys. B* **473**[FS22] (1988).
14. H. W. Blöte, J. L. Cardy, and M. P. Nightingale, *Phys. Rev. Lett.* **56**:742 (1986); I. Affleck, *Phys. Rev. Lett.* **56**:746 (1986).
15. J. L. Cardy, *Phase Transitions and Critical Phenomena*, Vol. 11, C. Domb and J. L. Lebowitz, eds. (Academic Press, New York, 1987).
16. F. C. Alcaraz, M. Baake, U. Grimm, and V. Rittenberg, *J. Phys. A* **22**:L5 (1989); F. C. Alcaraz, U. Grimm, and V. Rittenberg, *Nucl. Phys.*, to appear.
17. D. Friedan, Z. Qiu, and S. Shenker, *Phys. Rev. Lett.* **52**:1575 (1984).
18. J. Babujian, *Phys. Lett.* **90A**:479 (1982); L. Takhtajan, *Phys. Lett.* **87A**:479 (1982).
19. F. C. Alcaraz and M. J. Martins, *Phys. Rev. Lett.* **61**:1529 (1988); *J. Phys. A*, in press.
20. A. B. Zamolodchikov and V. A. Fateev, *Sov. J. Nucl. Phys.* **32**:2 (1980); K. Sogo, *Phys. Lett.* **104A**:51 (1984).
21. F. Vallet, R. Schilling, and S. Aubry, *Europhys. Lett.* **2**:815 (1986).